8. If the equation is of the first degree with respest to p, and algebraic with respect to x and y, p will be possible for every point of the plane and there will be no singular solution. If the equation is algebraic and of a degree higher than the second with respect to p we may apply the usual condition for equal roots that is to say, the equations $\varphi(x, y, p)=0$ and $\varphi'(x, y, p)=0$, where $\varphi'=\dfrac{d\varphi}{dp}$ must be satisfied by a common value of p: hence eliminating p between these equations we have the condition expressed as a relation between x and y. For example, given

$$\varphi=p^3-4xyp+8y^2=0,$$

then

$$\varphi'=3p^2-4xy=0;$$

eliminating p we find $y=0$, and $27y=4x^3$.

Each of these is a branch of the envelop of the complete primitive, and is a singular solution. The complete primitive in fact is $y=c(x-c)^2$, representing a series of parabolas which touch the axis of x and the cubical parabola $27y=4x^3$.

SOME TRIGONOMETRIC SERIES.

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1. TAKE the equations

$$\log(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4,\ldots \quad (1)$$

$$\log(1-x)=-x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\tfrac{1}{4}x^4,\ldots \quad (2)$$

$$\theta\sqrt{-1}=\varphi,\ 2\cos\theta=e^{\phi}+e^{-\phi},\ 2\sqrt{-1}.\sin\theta=e^{\phi}-e^{-\phi},\ldots\ldots \quad (3)$$

$$\log 2\cos\theta=\varphi+\log(1+e^{-2\phi})=-\varphi+\log(1+e^{2\phi}),\ \ldots\ldots \quad (4)$$

$$\log 2\sin\theta=-\log\sqrt{-1}+\varphi+\log(1-e^{-2\phi})=\log\sqrt{-1}-\varphi+\log(1-e^{2\phi}), \quad (5)$$

$$2\log 2\cos\theta=\log(1+e^{2\phi})+\log(1+e^{-2\phi}),\ \ldots\ldots \quad (6)$$

$$2\log 2\sin\theta=\log(1-e^{2\phi})+\log(1-e^{-2\phi}).\ \ldots\ldots \quad (7)$$

Developing (6) and (7) by (1) and (2), we have

$$2\log 2\cos\theta=e^{2\phi}+e^{-2\phi}-\tfrac{1}{2}(e^{4\phi}+e^{-4\phi})+\tfrac{1}{3}(e^{6\phi}+e^{-6\phi}),\ldots (8)$$

$$2\log 2\sin\theta=-(e^{2\phi}+e^{-2\phi})-\tfrac{1}{2}(e^{4\phi}+e^{-4\phi})-\tfrac{1}{3}(e^{6\phi}+e^{-6\phi}).\ldots (9)$$

Whence we have

$$\log\ 2\cos\theta=\cos 2\theta-\tfrac{1}{2}\cos 4\theta+\tfrac{1}{3}\cos 6\theta-,\ \ldots (10)$$

$$\log(2\sin\theta)^{-1}=\cos 2\theta+\tfrac{1}{2}\cos 4\theta+\tfrac{1}{3}\cos 6\theta+\ldots (11)$$

By differentiating (10) we have

$$\tan\theta=2[\sin 2\theta-\sin 4\theta+\sin 6\theta-\ldots]\ \ldots\ (12)$$

2. From (4) we have $2\varphi = 2\theta\sqrt{-1} = \log(1+e^{2\phi}) - \log(1+e^{-2\phi})$

$$= (e^{2\phi} - e^{-2\phi}) - \tfrac{1}{2}(e^{4\phi} - e^{-4\phi}) + \ldots \quad (13)$$

Whence we have $\theta = \sin 2\theta - \frac{1}{2}\sin 4\theta + \frac{1}{3}\sin 6\theta - \ldots$ (14)

Multiply this by $d\theta$ and integrate, multiply again and integrate, and continue this process. Denote the arbitrary constants by C_2, C_4, C_6, &c., noticing that when the sine occurs in the series, this constant is zero. We shall have, if $1.2.3.\ldots n = [n]$,

$$\frac{\theta^2}{[2]} + C_2 = -\frac{1}{2}\left[\cos 2\theta - \frac{\cos 4\theta}{2^2} + \frac{\cos 6\theta}{3^2}\ldots\right],$$

$$\frac{\theta^4}{[4]} + \frac{C_2\theta^2}{[2]} + C_4 = \frac{1}{2^3}\left[\cos 2\theta - \frac{\cos 4\theta}{2^4} + \frac{\cos 6\theta}{3^4}\ldots\right],$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{\theta^n}{[n]} + \frac{C_2\theta^{n-2}}{[n-2]} + \frac{C_4\theta^{n-4}}{[n-4]} + \ldots C_n = \frac{(-1)^{\frac{n}{2}}}{2^{n-1}}\left[\cos 2\theta - \frac{\cos 4\theta}{2^n} + \frac{\cos 6\theta}{3^n}\ldots\right] \quad (15)$$

Now make $S_n = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \ldots$ (16), $S'_n = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n}\ldots$ (17)

$S''_n = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n}\ldots$ (18), $S'_n - S_n = \frac{1}{2^{n-1}}\left[1 + \frac{1}{2^n} + \frac{1}{3^n}\ldots\right] = \frac{1}{2^{n-1}}S'_n$;

$$S'_n = S_n.\frac{2^{n-1}}{2^{n-1}-1}\ldots (19),\quad S''_n = \tfrac{1}{2}(S'_n + S_n) = S_n.\frac{2^n - 1}{2^n - 2}\ldots (20)$$

Now make $\theta = \frac{1}{4}\pi = p$ in (15), and also make $\theta = 0$, and we find

$$\frac{p^n}{[n]} + \frac{C_2 p^{n-2}}{[n-2]} + \frac{C_4 p^{n-4}}{[n-4]} + \ldots + C_n = \frac{(-1)^{\frac{1}{2}n}}{2^{2n-1}}S_n \ldots (21);\quad C_n = \frac{(-1)^{\frac{1}{2}n}}{2^{n-1}}S_n \ldots (22)$$

In these equations n is even. In (22) make $n = 2, 4, 6$, &c., and substitute in (21) and we have

$$\frac{p^n}{[n]} - \frac{S_2 p^{n-2}}{2.[n-2]} + \frac{S_4 p^{n-4}}{2^3[n-4]} - \frac{S_6 p^{n-6}}{2^5[n-6]} + \ldots + \frac{(-1)^{\frac{1}{2}n}}{2^{n-1}}S_n = \frac{(-1)^{\frac{1}{2}n}}{2^{2n-1}}.S_n \ldots (23)$$

From the form of (23) we easily see that p^n is a factor of all its terms; let us therefore make $\frac{1}{2^{n-1}}S_n = p^n B_n$, and we shall have

$$1 - n(n-1)B_2 + n(n-1)(n-2)(n-3)B_4 \ldots + (-1)^{\frac{1}{2}n}[n]B_n.\frac{2^n-1}{2^n} = 0 \ldots (24)$$

Now let $n = 2, 4, 6$, &c., in succession, and we shall have

$$B_2 = \tfrac{2}{3},\ B_4 = \tfrac{14}{45},\ B_6 = \tfrac{124}{945},\ \text{\&c.} \ldots (25)$$

We also have $S_n = 2^{n-1}p^n B_n = \frac{1}{2^{n+1}}\pi^n B_n$, $S'_n = \frac{\pi^n B_n}{2^{n+1}-4}\ldots$ (26)

Whence we find $S_2 = \frac{1}{12}\pi^2$, $S'_2 = \frac{1}{6}\pi^2$, $S''_2 = \frac{1}{8}\pi^2$;
$S_4 = \frac{7}{720}\pi^4$, $S'_4 = \frac{1}{90}\pi^4$, $S''_4 = \frac{1}{96}\pi^4$

We have the following well-known relation:

$$\log\cos\theta = \log\left(1-\frac{2^2\theta^2}{\pi^2}\right) + \log\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right) + \log\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right). \quad . \quad . \quad (27)$$

Developing by (2) we have

$$\log\cos\theta = -\frac{2^2\theta^2}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right)-\frac{2^4\theta^4}{2\pi^2}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\ldots\right)$$

$$-\frac{2^6\theta^6}{3\pi^6}\left(1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}\ldots\right)\ldots = -\frac{2^2\theta^2}{\pi^2}S''_2-\frac{2^4\theta^4}{2\pi^4}S''_4-\frac{2^6\theta^6}{3\pi^6}S''_6-\ldots \quad . \quad . \quad (28)$$

If we differentiate (28) we shall have

$$\tan\theta = \frac{2^3\theta}{\pi^2}S''_2 + \frac{2^5\theta^3}{\pi^4}S''_4 + \frac{2^7\theta^5}{\pi^6}S''_6 + \frac{2^9\theta^7}{\pi^8}S''_8 + \ldots; \quad . \quad . \quad . \quad (29)$$

$$\text{or} \quad 2\tan\theta = \frac{2^2-1}{2-1}B_2\theta + \frac{2^4-1}{2^3-1}B_4\theta^3+\frac{2^6-1}{2^5-1}B_6\theta^5+\frac{2^8-1}{2^7-1}B_8\theta^7+\ldots \quad (30)$$

3. Let us resume Eq. (12) and make

$$\tan\psi = 2[\sin 2\psi-\sin 4\psi+\sin 6\psi-\ldots] = A_1\psi+A_2\psi^2+A_3\psi^3+A_4\psi^4\ldots(31)$$

If we differentiate and make $\psi = 0$ after each differentiation, we shall find

$$A_1 = 2^2[1-2+3-4+\ldots],\ A_2 = 0,\ A_4 = 0,\ \&c.$$

$$[3]\,A_3 = -2^4[1-2^3+3^3-4^3+\ldots],$$

$$[5]\,A_5 = 2^6[1-2^5+3^5-4^5+\ldots],\ \&c.$$

Now make $1-2^{2n-1}+3^{2n-1}-4^{2n-1}+\ldots = (-1)^{n+1}\Sigma(1)^{2n-1} \ldots . (32)$
then we shall have

$$\tan\psi = 2^2\Sigma(1).\psi + \frac{2^4\Sigma(1)^3}{[3]}\psi^3 + \frac{2^6\Sigma(1)^5}{[5]}\psi^5 + \ldots \ldots (33)$$

We mnst now find the various sums represented by Σ. If we differentiate (14) we have $1 = 2[\cos 2\psi - \cos 4\psi + \cos 6\psi - \ldots]$

$$0 = \sin 2\psi - 2\sin 4\psi + 3\sin 6\psi - \ldots$$

$$0 = \cos 2\psi - 2^2\cos 4\psi + 3^2\cos 6\psi - \ldots \quad \&c.$$

Make $\psi = 0$ and we see that $0 = 1 - 2^{2n} + 3^{2n} - 4^{2n} + \ldots \ldots \ldots (34)$

Now let $u = (1+e^x)^{-1} = 1 - e^x + e^{2x} - e^{3x} + \ldots, \ldots \ldots (35)$
e being the Napierian base of logarithms. If we differentiate (35) and make $\frac{du}{dx} = u_1$, $\frac{d^2u}{dx^2} = u_2$, $\frac{d^3u}{dx^3} = u_3$, &c. and retain the same notation when we make $x = 0$, we shall have $u = \frac{1}{2}$, $u_1 = -\Sigma(1) = -\frac{1}{4}$, $u_2 = -\Sigma(1)^2 = 0$, by (34), $u_3 = \Sigma(1)^3$, $u_4 = 0$, $u_5 = -\Sigma(1)^5$, &c.

From (35) we have $ue^x + u = 1$. $\therefore$ $e^x(u+u_1)+u_1 = 0$, $e^x(u+2u_1+u_2)+u_2 = 0$, and generally

$$e^x\left[u+nu_1+\frac{n(n-1)}{[2]}u_2+\frac{n(n-1)(n-2)}{[3]}u_3+\ldots+nu_{n-1}+u_n\right]+u_n=0$$

If we make $x = 0$, and remember that u_2, u_4, &c, $= 0$, we shall have

$$u+nu_1+\frac{n(n-1)(n-2)}{[3]}u_3+\ldots+2u_n=0, \quad \ldots\ldots \quad (36)$$

in which n is an odd number. Or we may make

$$\tfrac{1}{2}=n\Sigma(1)-\frac{n(n-1)(n-3)}{[3]}\Sigma(1)^3+\ldots+(-1)^{\frac{1}{2}(n-1)}2\Sigma(1)^n \ldots (37)$$

If we now make $n=1, 3, 5$, &c., in succession, we shall have

$$\Sigma(1)=\tfrac{1}{4},\ \Sigma(1)^3=\tfrac{1}{8},\ \Sigma(1)^5=\tfrac{1}{4},\ \Sigma(1)^7=\tfrac{17}{16},\ \Sigma(1)^9=\tfrac{31}{4},\ \Sigma(1)^{11}=\tfrac{691}{8}, \text{ \&c.} \ldots (38)$$

These values i (33) give

$$\tan\phi=\phi+\frac{2^4}{[3]}.\tfrac{1}{8}\phi^3+\frac{2^6}{[5]}.\tfrac{1}{4}\phi^5+\frac{2^8}{[7]}.\tfrac{17}{16}\phi^7+\frac{2^{10}}{[9]}.\tfrac{31}{4}\phi^9+\frac{2^{12}}{[11]}.\tfrac{691}{8}\phi^{11}+\ldots (39)$$

By comparing (30) and (33) we see that

$$\frac{2^2-1}{2-1}.B_2=2,\ \frac{2^4-1}{2^3-1}.B_4=\frac{2^5}{[3]}\Sigma(1)^3, \ldots \frac{2^{2n}-1}{2^{2n-1}-1}B_{2n}=\frac{2^{2n+1}}{[2n-1]}\Sigma(1)^{2n-1}. \quad \ldots (40)$$

We readily see that $u_1=u^2-u$, and hence

$u_2=(2u-1)u_1=u(u-1)(2u-1)=2u^3-3u^2+u$, and generally

$$u_n=A_{n+1}u^{n-1}-A_nu^n+A_{n-1}u^{n-1}-\ldots+(-1)^nu, \quad \ldots\ldots (41)$$

$$\begin{aligned}u_{n+1}&=[(n+1)A_{n+1}u^n-nA_nu^{n-1}+(n-1)A_{n-1}u^{n-2}-\ldots+(-1)^n](u^2-u)\\&=(n+1)A_{n+1}u^{n+2}-[nA_n+(n+1)A_{n+1}]u^{n+1}+[(n+1)A_{n-1}+nA_n]u^n\\&\quad-\ldots+(-1)^{n+1}u. \quad \ldots (42)\end{aligned}$$

By means of (41) and (42) we can easily compute the values of u_2, u_3, &c., knowing the value of u_1. We see that the sum of the coefficients of (42) is 0.

Since $2u=1$, we may make $2u=z=1$, and then

$$2^{n+1}u_n=A_{n+1}z^{n+1}-2A_nz^n+2^2A_{n-1}z^{n-1}-\ldots+2^n(-1)^nz, \quad \ldots\ldots (43)$$

$$2^{n+2}u_{n+1}=(n+1)A_{n+1}z^{n+2}-2[nA_n+(n+1)A_{n+1}]z^{n+1}+\ldots+2^{n+1}(-1)^{n+1}z \quad \ldots (44)$$

Now let us make $2^{p+1}A_{n-p}=V_{n-p} \ldots (45)$, then, if $2^{n+1}u_n=z_n$, ..(46) we shall have $z_n=V_{n+1}z^{n+1}-V_nz^n+V_{n-1}z^{n-1}-\ldots+2^n(-1)^nz$, ..(47)

$$z_{n+1}=(n+1)V_{n+1}z^{n+2}-[nV_n+2(n+1)V_{n+1}]z^{n+1}+[(n-1)V_{n-1}+2nV_n]z^n-\ldots+2^{n+1}(-1)^{n+1}z. \quad \ldots (48)$$

Since $z=1$, $\quad z_n=V_{n+1}-V_n+V_{n-1}-\ldots+2^n(-1)^n, \quad \ldots\ldots (49)$

$$z_{n+1}=-(n+1)V_{n+1}+nV_n-(n-1)V_{n-1}+\ldots+2^n(-1)^{n+1} \quad \ldots (50)$$

If we now make $n=1$, we have $V_2=1$, $V_1=2$, and

$$\begin{aligned}z_1&=z^2-2z&&=1-2&&=-1,\\z_2&=2z^3-6z^2+4z&&=-2+2&&=0,\\z_3&=6z^4-24z^3+28z^2-8z&&=-6+12-4&&=2,\\z_4&=24z^5-120z^4+200z^3-120z^2+16z&&=-24+72-56+8&&=0.\end{aligned}$$

These equations are easily formed. We find $u_1 = \frac{1}{4}z_1 = -\frac{1}{4}$, $u_3 = \frac{1}{16}z_3 = \frac{1}{8}$, &c. It will be noticed that $\Sigma(1)^{2n+1}$ has some power of 2 for a denominator, and the relation of these numbers to the *Nnmbers of Bernoulli*, is easily ascertained.

If again we make $1.2.3\ldots n = [n]$, I have found

$$z_n = [n]z^{n+1} - [n+1]z^n + \tfrac{1}{6}[n+1](3n-2)z^{n-1} - \tfrac{1}{6}[n+1](n-1)(n-2)z^{n-2}$$
$$+ \tfrac{1}{360}[n+1](15n^3 - 105n^2 + 230n - 152)z^{n-3} - \ldots \quad (51)$$

The remaining coefficients I have not yet obtanied.

If we take

$$1 + x + x^2 + x^3 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1},$$

and make $x - 1 = y$, then $dx = dy$ and

$$1 + x + x^2 + \ldots + x^n = \frac{(1+y)^{n+1} - 1}{y} = n + 1 + \frac{n(n+1)}{1.2}y$$
$$+ \frac{n(n+1)(n-1)}{1.2.3}y^2 + \ldots (1)$$

If we differentiate and make $\frac{n(n+1)}{1.2} = N_1$, $\frac{n(n+1)(n-1)}{1.2.3} = N_2$, &c., we shall have

$$1 + 2x + 3x^2 + \ldots + nx^{n-1} = N_1 + 2N_2y + 3N_3y^2 + \ldots \ldots \quad (2)$$

Multiply by $x = 1 + y$, and make $A_1 = N_1 + 2N_2$, $A_2 = 2N_2 + 3N_3$, &c., and we shall have

$$x + 2x^2 + 3x^3 + \ldots + nx^n = N_1 + A_1y + A_2y^2 + \ldots \ldots \quad (3)$$

Differentiate again and we have

$$1 + 2^2x + 3^2x^2 + \ldots + n^2x^{n-1} = A_1 + 2A_2y + 3A_3y^2 + \ldots$$

Again multiply by $x = 1 + y$, put $A_1 + 2A_2 = B_1$, $2A_2 + 3A_3 = B_2$, $3A_3 + 4A_4 = B_3$, &c., and continue the process, and then make $x = 1$, and $y = 0$ in these equations, and we shall have, if

$$S_n^{(p)} = 1 + 2^p + 3^p + \ldots + n^p,$$
$$S_n^{(1)} = N_1,$$
$$S_n^{(2)} = A_1 = N_1 + 2N_2,$$
$$S_n^{(3)} = B_1 = A_1 + 2A_2,$$
$$S_n^{(4)} = C_1 = B_1 + 2B_2,$$
&c. &. &c.
$$A_2 = 2N_2 + 3N_3,$$
$$B_2 = 2A_2 + 3A_3,$$
&c. &c.
$$A_3 = 3N_3 + 4N_4,$$
&c. &c.